

Scattering Equations and String Theory Amplitudes

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ABSTRACT: Scattering equations for tree-level amplitudes are viewed in the context of string theory. As a result of the comparison we are led to define a new dual model which coincides with string theory in both the small and large α' limit, and whose solution is found algebraically on the surface of solutions to the scattering equations. Because it has support only on the scattering equations, it can be solved exactly, yielding a simple resummed model for α' -corrections to all orders. We use the same idea to generalize scattering equations to amplitudes with fermions and any mixture of scalars, gluons and fermions. In all cases checked we find exact agreement with known results.

KEYWORDS: Scattering amplitudes, Scattering equations, String theory.

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1. Introduction

In a series of remarkable papers, Cachazo, He and Yuan have proposed that tree level scattering of massless particles in any dimension can be constructed from kinematic solutions of a set of algebraic scattering equations [1–3]. This idea has been supported by a number of highly non-trivial observations and checks, and also by explicit amplitude computations for a large number of external legs. A proof of this surprising construction has recently been provided for scalar amplitudes and gluon amplitudes by Dolan and Goddard in ref. [4] based on BCFW [5] recursion. These authors have also shown how to generalize the construction to massive scalars, extending the specific construction for scalars of ref. [3] to any theory of scalars with only 3-point vertices, again in any dimension.

The whole setup of the scattering equation approach is eerily reminiscent of string theory, and indeed it was recognized early on [2] that these scattering equations coincide with the saddle point of the Gross-Mende limit [6]. But this also represents a conundrum: The Gross-Mende limit is that of high energy scattering of strings corresponding to $\alpha' \rightarrow \infty$, not the opposite limit of $\alpha' \rightarrow 0$ where the field theory of point-like particles emerges. Indeed, in the $\alpha' \rightarrow 0$ limit of string theory an entirely different formalism arises, even though, eventually, the same tree-level amplitudes

come out. It is as if the scattering equation approach has managed to obtain a different limit of $\alpha' \rightarrow 0$, while retaining aspects of high-energy scattering of strings. In the twistor string frameworks [7–10], it has been demonstrated that one can naturally impose the scattering equations in an alternative path integral formulation. Many other indications of a close connection to string theory can be found. In [1] it was thus shown that the scattering equations are intimately related to the momentum kernel S [11, 12] between gauge and gravity theories (and hence between open and closed strings). Similarly, scattering equations manifestly operate with a basis of $(N - 3)!$ amplitudes, in agreement with what is inferred from BCJ-relations [13] and which follows directly from string theory [14, 15]. A more direct link between BCJ-relations and scattering equations have also been proposed [3, 16, 17]. Finally, some algebraic relations arising in the string theory computation of disk amplitudes [18–20] have also found use in the scattering equation formalism. All of these examples indicate a close connection to string theory.

In this paper we suggest a new approach which takes as starting point the observation that the integrand of string theory amplitudes, through a series of partial integrations, can be given a form that achieves two objectives simultaneously: (i) it reproduces all string theory tree-level amplitudes for any α' when integrated over the usual string theory integration measure (by construction, since it is obtained from the conventional form only through the use of integration by parts) and (ii) it reproduces the tree-level amplitudes of field theory in the $\alpha' \rightarrow 0$ limit even when integrated over the $\text{SL}(2, \mathbb{C})$ invariant measure of Cachazo, He and Yuan. Interestingly, this latter observation leads us to the conclusion that it is straightforward and natural to introduce (ii) a new dual model which gives field theory amplitudes back in the $\alpha' \rightarrow 0$ limit and which in the limit of $\alpha' \rightarrow \infty$ reproduces the high-energy limit of string scattering. At intermediate values of α' it is not a string theory. Remarkably, nevertheless, it agrees exactly with string theory in the two limits mentioned. We view it as a new dual model which could have been introduced long ago. Indeed, the approach by Fairlie et al. [21] (reviewed in [22]) by imposing on a scalar dual model a minimal area constraint is closely related to this, only missing the more general context and the new connection to the field theory limit $\alpha' \rightarrow 0$ that we provide here.

To show that the approach we suggest here also can be used to derive new results, we illustrate how the formalism (ii) can be extended to include fermions as in the superstring. We demonstrate explicitly that this produces correct amplitudes with fermions in a few simple cases. Also examples of mixed amplitudes with scalars, gluons and fermions will be shown.

Our paper is organized in the following way. First, we briefly review the scattering equations and their solution in the field theory limit. Next, we introduce a simple dual model, tachyon-free, of scalars. By imposing on the integrand the scattering equations, we obtain a simple scalar analog of the general framework of this

paper: a model that reproduces the field theory limit on the surface of solutions to the scattering equations as $\alpha' \rightarrow 0$ and which reproduces the Gross-Mende solution in the limit of $\alpha' \rightarrow \infty$. By itself, such a definition of a new dual model is rather trivial and ad hoc. That there is more to the story becomes evident when we next turn to amplitudes involving gauge fields. We first briefly recall how to compute the corresponding gluon amplitudes in string theory. As is well known, the resulting expression is rather cumbersome, but it can be rearranged into a form identical to the Pfaffian prescription of refs. [1–3], up to additional terms that formally are suppressed as $1/\alpha'$ in the integrand¹. We next investigate under which manipulations we can absorb the additional terms into the Pfaffian of [1–3]. The simple observation is that all the additional pieces are proportional to the scattering equations after suitable integrations by parts. Performing these partial integrations in the string theory measure, the resulting modified integrand yields (i) all string theory amplitudes when integrated over the string theory measure and (ii) the correct field theory limit $\alpha' \rightarrow 0$ when localized to the surface of solutions to the scattering equations. This elementary observation is particularly easily understood in the case of the four-point gluon amplitude. We also show how such manipulations extend to higher point amplitudes. The key feature here is that integration-by-part relations in the string theory measure yield terms proportional the scattering equations. On the surface of solutions to these equations they do not contribute, which is why they play no role in our new dual model (ii). Moreover, the partial integrations are precisely those that provide a manifest cancellation of tachyonic terms in the integrand. Finally, in section six we discuss how to extend these considerations to compute amplitudes with external fermions on the basis of the scattering equations, and how mixed amplitudes with scalars, fermions and vectors can be computed as well. We end with an outlook for future work.

2. Scattering Equations and a Dual Model Extension

For scalar theories, the prescription given by Cachazo, He and Yuan for computing an N -point amplitude reads

$$A_N = \int \prod'_i \delta(S_i) \frac{(z_1 - z_{N-1})(z_1 - z_N)(z_{N-1} - z_N)}{\prod_{i=1}^N (z_i - z_{i+1})^2} \prod_{i=2}^{N-1} dz_i. \quad (2.1)$$

Here S_i denotes the i th scattering equation

$$S_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0. \quad (2.2)$$

¹But as will become evident shortly, such a naïve counting of powers of α' in the integrand does not match the corresponding counting in the amplitude after the integrals have been performed.

In the following we will fix the three points $z_1 = 0$, $z_{N-1} = 1$ and $z_N = \infty$.

Let us now try to see this construction in the light of old-fashioned dual models with a dimensionful parameter α' . A simple dual model that yields the same massless scalar scattering amplitudes in the limit $\alpha' \rightarrow 0$ is the following:

$$\mathcal{A}_N = \left(\frac{g}{2}\right)^{N-2} (\alpha')^{N-3} \int \left(\prod_{i=1}^N dz_i \right) / d\omega \prod_{1 \leq i < j \leq N} |z_i - z_j|^{2\alpha' k_i \cdot k_j} \prod_{i=1}^N (z_i - z_{i+1})^{-1}, \quad (2.3)$$

where the integration is ordered along the real axis and we define

$$d\omega = \frac{dz_1 dz_{N-1} dz_N}{(z_1 - z_{N-1})(z_{N-1} - z_N)(z_N - z_1)}, \quad (2.4)$$

and the notation is such that $z_{N+1} \equiv z_1$.

Note how different the integration prescription is in the two cases. In the simple dual model defined above, we integrate in an ordered manner along the real line after having fixed again $z_1 = 0$, $z_{N-1} = 1$ and $z_N = \infty$. In the integral defining amplitudes based on scattering equations (2.1) the integral is saturated by the solutions to the δ -function constraints. This means that singularities that normally carry the whole amplitude in the $\alpha' \rightarrow 0$ limit are harmless. Also the remaining part of the integrand is of course totally different, as there is no trace of α' in (2.1). Yet, remarkably, for all N the $\alpha' \rightarrow 0$ limit of (2.3) yields exactly the same answer as (2.1). This suggests that it may be advantageous to view (2.1) as the leading term of a more elaborate amplitude that depends on a parameter α' . Indeed, the only argument for choosing the $SL(2, C)$ -invariant integration measure of (2.3) is historical.

Based on this perhaps rather naïve argument, let us introduce a very simple new dual model defined by amplitudes

$$\begin{aligned} \mathcal{A}_N &= \left(\frac{g}{2}\right)^{N-2} \int \left(\prod_{i=1}^N dz_i \right) \frac{1}{d\omega} (z_1 - z_{N-1})(z_1 - z_N)(z_{N-1} - z_N) \\ &\quad \prod_{1 \leq i < j \leq N} |z_i - z_j|^{2\alpha' k_i \cdot k_j} \prod_i' \delta(S_i) \prod_{i=1}^N (z_i - z_{i+1})^{-1} \prod_{i=1}^N (z_i - z_{i+1})^{-1} \\ &= \left(\frac{g}{2}\right)^{N-2} \int \prod_{i=2}^{N-2} dz_i \prod_{1 \leq i < j \leq N-2} |z_i - z_j|^{2\alpha' k_i \cdot k_j} \prod_i' \delta(S_i) \prod_{i=1}^{N-2} (z_i - z_{i+1})^{-2}. \end{aligned} \quad (2.5)$$

Note that, effectively, this simply amounts to taking the dual model expression and inserting the normalized δ -function constraint²

$$(z_1 - z_{N-1})(z_1 - z_N)(z_{N-1} - z_N) \prod_i' \delta(S_i) \prod_{i=1}^N (z_i - z_{i+1})^{-1}, \quad (2.6)$$

²In general, this can be given a precise interpretation in terms of contours in the complex plane via the global residue theorem [2, 4]. However in all cases we have considered (even in the case of complex solutions to the scattering equations) the naïve δ -function constraint works as well, and of course the final result is real.

in the integrand. This is what we will claim is the general prescription, also after we turn to string theory.

Massive scalar amplitudes can be dealt with easily, as they simply correspond to replacing

$$\prod_{i=1}^N (z_i - z_{i+1})^{-1} \rightarrow \prod_{i=1}^N (z_i - z_{i+1})^{-1-\alpha' m^2} ,$$

in the integrand of (2.3). By differentiation of the integrand with respect to z_i we obtain the massive scattering equation proposed and proven to be correct in ref. [4]. The fact that scattering equations arise from differentiation with respect to the z_i of external legs in the integrand will play a crucial role in what follows.

In contrast to a more conventional dual model such as (2.3), the new integral (2.5) has a totally smooth and finite limit $\alpha' \rightarrow 0$, where it of course coincides with scalar field theory. So has anything been achieved in making such a trivial extension? A hint that this may be so is that in the opposite limit $\alpha' \rightarrow \infty$, the amplitudes (2.5) coincide with those of (2.3), where both indeed coincide with the Gross-Mende limit of high energy string scattering. So this simple extension (2.5) retains all the nice properties of (2.1) when $\alpha' = 0$, and yields stringy amplitudes in the opposite limit of $\alpha' \rightarrow \infty$. In between these two limits we obviously have no immediate way to interpret the amplitudes (2.5), but these amplitudes are all trivially computable due to the δ -function constraint in the measure.

What could be the meaning of the dimensional parameter α' here? It would be tempting to view it as an inverse string tension. However, such a point of view is not tenable. This becomes clear already in the case of four-particle scattering, which has almost no resemblance at finite α' to the corresponding Veneziano amplitude of (2.3). There is not an infinite series of poles in the amplitude which, rather, is more like that of ordinary field theory with a trivial exponential damping factor. Indeed, because the limit $\alpha' \rightarrow 0$ meets no singularity, amplitudes with either small or large momenta can be found immediately at any value of α' . At $\alpha' = 0$ the scattering amplitudes of (2.5) are just those of field theory, up to arbitrarily high energies. The extension of (2.1) to the new dual model (2.5) looks much like dualized (color-ordered) scalar field theory regularized with an ultraviolet cut-off $1/\sqrt{\alpha'}$.

At this point, the dual model (2.5) therefore cannot be viewed as more than a curiosity. If there is to be any substance in it, and insight to be gained, we must see if a slightly more sophisticated line of approach can yield new results. We therefore turn to ordinary string theory, and explore the extent to which similar considerations can be extended to massless gauge boson scattering.

3. Scattering Equations and Gauge Fields

The prescription given in [1–3] for computing an N -point gauge theory amplitude

reads

$$A_N = \int \text{Pf}' \Psi_N(z_i) \prod_i' \delta(S_i) \prod_{i=1}^N \frac{1}{(z_i - z_{i+1})} \prod_{i=2}^{N-2} dz_i, \quad (3.1)$$

where S_i denotes the i th scattering equation as in the scalar case. The function $\text{Pf}' \Psi_N(z_i)$ is given via

$$\Psi_N(z_i) = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad (3.2)$$

where

$$A_{i,j} = \begin{cases} \frac{k_i \cdot k_j}{z_i - z_j} & i \neq j, \\ 0 & i = j, \end{cases} \quad B_{i,j} = \begin{cases} \frac{\epsilon_i \cdot \epsilon_j}{z_i - z_j} & i \neq j, \\ 0 & i = j, \end{cases} \quad C_{i,j} = \begin{cases} \frac{\epsilon_i \cdot k_j}{z_i - z_j} & i \neq j, \\ -\sum_{l \neq i} \frac{\epsilon_i \cdot k_l}{z_i - z_l} & i = j. \end{cases} \quad (3.3)$$

From $\Psi_N(z_i)$ we can define the reduced Pfaffian, $\text{Pf}'(\Psi)$ by

$$\text{Pf}' \Psi = \frac{(-1)^{i+j}}{z_i - z_j} \text{Pf}(\Psi_{ij}^{ij}), \quad (3.4)$$

where Ψ_{ij}^{ij} is the matrix obtained from Ψ by removing the rows and columns i and j (2 rows and 2 columns removed).

4. The Pfaffian Form of String Amplitudes with Gluons

It is striking how similar the prescription of the previous section is to that of string theory. In this section we explore this in detail. It is well known that the requirement of multi-linearity in external polarization vectors conveniently can be implemented in terms of auxiliary fermionic integrations in the string integrand. These real Grassmann variables, when integrated out, produce a Pfaffian. This suggests that the Pfaffian prescription of the previous section may be viewed as a remnant of the string theory integrand, now only evaluated on the solutions to the scattering equations. As we shall see, this is indeed the case. But instead of computing the resulting Pfaffian directly, it is convenient to split it up into its separate components, in this way illuminating which pieces give rise to the Pfaffian of the previous section, and which do not.

4.1 Multi-Pfaffian Structure of N -point Open-String Integrand.

In this section we provide another way to decompose the string theory integrand for the scattering of N gluons in the open superstring as a sum of Pfaffians. This will include terms in the integrand of increasing powers of $1/\alpha'$ as N grows, but of course the full integral starts with terms of order $1/\alpha'$ only. These terms of higher powers of $1/\alpha'$ in the integrand can indeed be re-cast into terms that carry no explicit factor

of α' by means of integrations by parts. Such rewritings show that these terms do not contribute on the surface of solutions to the scattering equations.

In the RNS formalism, the vertex operators come in various ghost-pictures with respect to the superconformal ghost ($\beta = \partial\xi e^{-\varphi}, \gamma = e^{\varphi}$) [23]. The (-1) -picture of the unintegrated vertex operator for the emission of a gauge boson is then given by

$$U^{(-1)} = g_o T^a : e^{-\varphi} \epsilon \cdot \psi e^{ik \cdot X} : , \quad (4.1)$$

while these in the (0) -ghost picture read

$$U^{(0)} = g_o \sqrt{\frac{2}{\alpha'}} T^a : (i\partial X^\mu + 2\alpha'(k \cdot \psi)(\epsilon \cdot \psi)) e^{ik \cdot X} : . \quad (4.2)$$

The corresponding integrated vertex operators are given by

$$\begin{aligned} V^{(-1)} &= \int dz : U^{(-1)} : , \\ V^{(0)} &= \int dz : U^{(0)} : . \end{aligned} \quad (4.3)$$

The normalization of the OPE on the boundary of the disk is such that

$$X^\mu(z) X^\nu(0) \simeq -\alpha' \log |z|^2 , \quad (4.4)$$

$$\psi^\mu(z) \psi^\nu(0) \simeq \frac{\eta^{\mu\nu}}{z} , \quad (4.5)$$

$$e^{q_1 \varphi(z)} e^{q_2 \varphi(0)} \simeq \frac{1}{z^{q_1 q_2}} . \quad (4.6)$$

At tree-level, to saturate the $+2$ background superghost charge, one should set two vertex operators in the (-1) picture, the rest can be chosen in the (0) picture. These two operators chosen in the (-1) picture, for instance V_1 and V_2 , determine which lines and columns of the matrix one should remove to get the correct reduced Pfaffian. The n -gluon open string amplitude \mathcal{A}_N reads:

$$\mathcal{A}_N = \langle cU^{(-1)}(z_1) cU^{(0)}(z_{N-1}) cU^{(0)}(z_N) \int \prod_{i=2}^{N-2} dz_i V^{(-1)}(z_2) \cdots V^{(0)}(z_{N-2}) \rangle . \quad (4.7)$$

A Pfaffian comes out of this integral simply because of the Grassmann integral over a product of fermionic fields.

Focusing first on the purely fermionic part of the correlator (4.7), it involves a product of $2N - 2$ fermionic fields, among which $N - 3$ are bilinears:

$$\langle (\epsilon_1 \cdot \psi(z_1)) (\epsilon_2 \cdot \psi(z_2)) \prod_{i=3}^N : (k_i \cdot \psi(z_i)) (\epsilon_i \cdot \psi) : \rangle . \quad (4.8)$$

The integral

$$\int [d\psi] \epsilon_1 \cdot \psi(z_1)) (\epsilon_2 \cdot \psi(z_2)) \prod_{i=3}^N : (k_i \cdot \psi(z_i)) (\epsilon_i \cdot \psi) : \exp \left(-1/2 \int \psi \bar{\partial} \psi \right), \quad (4.9)$$

can therefore be written in terms of the following $(2N - 2) \times (2N - 2)$ matrix:

$$M' = \begin{pmatrix} A & -C'^T \\ C' & B \end{pmatrix}, \quad (4.10)$$

composed of the block matrices A , B and C' for which we have

$$\begin{aligned} A_{i,j} &= \frac{k_i \cdot k_j}{z_i - z_j}, & i, j &= 3, 4, \dots, N, \\ B_{i,j} &= \frac{\epsilon_i \cdot \epsilon_j}{z_i - z_j}, & i, j &= 1, 2, \dots, N, \\ C'_{i,i} &= 0, \quad C'_{ij} = \frac{\epsilon_i \cdot k_j}{z_i - z_j}, & i &= 1, 2, \dots, N, \quad j = 3, 4, \dots, N, \quad j \neq i. \end{aligned} \quad (4.11)$$

These matrices are of sizes $(N - 2) \times (N - 2)$, $N \times N$ and $N \times (n - 2)$, respectively, because the vertex operators corresponding to particles 1 and 2 do not have corresponding $k_i \cdot \psi$.

This is not yet the Pfaffian of eq. (3.4) because the matrix C' has 0's on the diagonal since the self-contraction $\langle : (k_i \cdot \psi(z_i)) (\epsilon_i \cdot \psi(z_i)) : \rangle$ vanishes. This self-contraction must be replaced by the bosonic contraction of a ∂X field with the plane-wave factor just as in ref. [7],

$$: (\epsilon_i \cdot \partial X(z_i)) e^{i \sum_l k_l X(z_l)} : \sim \left(-2\alpha' \sum_l \frac{\epsilon_i \cdot k_l}{z_i - z_l} \right) : e^{i \sum_{l \neq i} k_l X(z_l)} : + O(z_i - z_l), \quad (4.12)$$

providing the correct factor to add to the diagonal of the matrix C'

$$C_{i,i} = - \sum_l \frac{\epsilon_i \cdot k_l}{z_i - z_l}, \quad C_{ij} = C'_{ij}, \quad j \neq i, \quad (4.13)$$

and thus matching the Pfaffian of the matrix Ψ_{12}^{12} . After including the superghost correlator $\langle e^{\varphi_1} e^{\varphi_2} \rangle = z_{12}^{-1}$, we end up with $\text{Pf}' \Psi$ defined in (3.4).

In the approach of ref. [7] there are here no other contractions to perform because the ∂X field by construction is taken to be a momentum P field frozen by the scattering equations. However, here the story is different as we are here dealing with *actual string theory*. The ∂X fields do have non-vanishing OPE's with other ∂X fields. This is also the mechanism that prevent unwanted tachyon poles from appearing in the string theory amplitudes.

In order to derive these remaining terms, one can simply recursively apply Wick's theorem. In the first step, one finds OPE's only between ∂X 's and the plane-wave

factor; this gives the Pfaffian in eq. (3.4). In the second step, one performs all possible contractions between only two ∂X 's, the rest as before; this yields a sum of Pfaffians where two more sets of lines and rows have been crossed out, with a corresponding $\langle \partial X(z) \partial X(w) \rangle \sim (z-w)^{-2}$ propagator in front of it (this induces a weighing $1/\alpha'$ compared to the term of the first step). By iterating the process, one finally deduces that the chiral kinematic correlator is expressed as a sum of Pfaffians and the full answer is

$$\mathcal{A}_N = \int \prod_{i=2}^{N-2} dz_i \prod_{1 < i < j < N-1} |z_{ij}|^{2\alpha' k_i \cdot k_j} \times (z_{1,N-1} z_{N-1,n} z_{N1}) \times \left(\text{Pf}'(\Psi) + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{(2\alpha')^k} \sum_{\substack{\text{distinct pairs} \\ (i_3, i_4), \dots, (i_{2k-1}, i_{2k})}} \prod_{p=3}^{2k-1} \frac{(\epsilon_{i_p} \cdot \epsilon_{i_{p+1}})}{(z_{i_p i_{p+1}})^2} \text{Pf}'(\Psi_{i_3 i_4 \dots i_{2k}}^{i_3 i_4 \dots i_{2k}}) \right), \quad (4.14)$$

where a global normalization factor has been set to 1 and where $\text{Pf}'(\Psi_{i_3 i_4 \dots i_{2k}}^{i_3 i_4 \dots i_{2k}})$ stands for $\frac{1}{z_{12}} \text{Pf}(\Psi_{12 i_3 i_4 \dots i_{2k}}^{12 i_3 i_4 \dots i_{2k}})$.

We end this section with a few comments on closed string theory. If we turn to the heterotic string (see also ref. [7]), the same computations carry through and one obtains the following N -gluon amplitude:

$$\mathcal{A}_N^{\text{het}} = \int \prod_{i=2}^{N-2} d^2 z_i \prod_{1 < i < j < N-1} |z_{ij}|^{\alpha' k_i \cdot k_j} \times |z_{1,N-1} z_{N-1,N} z_{N1}|^2 \times \left(\frac{\text{Tr}(T^{a_1} \dots T^{a_N})}{\bar{z}_{12} \dots \bar{z}_{N1}} + \dots \right) \times \left(\text{Pf}'(\Psi) + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{(\alpha')^k} \sum_{\substack{\text{distinct pairs} \\ (i_3, i_4), \dots, (i_{2k-1}, i_{2k})}} \prod_{p=3}^{2k-1} \frac{(\epsilon_{i_p} \cdot \epsilon_{i_{p+1}})}{(z_{i_p i_{p+1}})^2} \text{Pf}'(\Psi_{i_3 i_4 \dots i_{2k}}^{i_3 i_4 \dots i_{2k}}) \right). \quad (4.15)$$

Once one considers closed string theory, it is natural to also consider gravity amplitudes which come in as a bonus. As recently shown [24, 25], one can transform the type II integral as a holomorphic square, which then produces the following formula for an N -graviton scattering

$$\mathcal{A}_N^{\text{II}} = \int \prod_{i=2}^{N-2} d^2 z_i \prod_{1 < i < j < N-1} |z_{ij}|^{\alpha' k_i \cdot k_j} \times |z_{1,N-1} z_{N-1,N} z_{N1}|^2 \times \left| \text{Pf}'(\Psi) + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{(\alpha')^k} \sum_{\substack{\text{distinct pairs} \\ (i_3, i_4), \dots, (i_{2k-1}, i_{2k})}} \prod_{p=3}^{2k-1} \frac{(\epsilon_{i_p} \cdot \epsilon_{i_{p+1}})}{(z_{i_p i_{p+1}})^2} \text{Pf}'(\Psi_{i_3 i_4 \dots i_{2k}}^{i_3 i_4 \dots i_{2k}}) \right|^2. \quad (4.16)$$

5. From String Theory to Scattering Equations

In the previous section we have seen which pieces of string theory give rise to the Pfaffian of eq. (3.4), and which yield additional terms. We will now show that the

additional terms, through partial integrations, can be put in a form that make them proportional to the scattering equations and hence vanishing on the alternative integration measure that imposes scattering equations as a δ -function constraint. In this form the full expression can be integrated over these two different measures, both yielding the correct field theory result when taking the $\alpha' \rightarrow 0$ limit. Some simple examples will illustrate this.

Let us for simplicity focus on the 4-point amplitude. As shown in the previous section, it takes the form

$$A_4(1, 2, 3, 4) = \int_0^1 \left(\text{Pf}'(\Psi) + \frac{(\epsilon_1 \epsilon_2)(\epsilon_3 \epsilon_4)}{2\alpha' z_{12}^2} \right) |z_1 - z_2|^{2\alpha' k_1 \cdot k_2} |z_2 - z_3|^{2\alpha' k_2 \cdot k_3} dz_2, \quad (5.1)$$

where as usual $z_1 = 0$, $z_3 = 1$ and $z_4 = \infty$. The additional piece proportional to $1/\alpha'$ is crucial in the string theory context, as it removes a tachyon pole and allows the limit $\alpha' \rightarrow 0$ to be taken, yielding the field theory answer.

One notices that the term

$$\delta A_4 = \int_0^1 dz_2 \frac{1}{z_{12}^2} \exp(2\alpha' k_1 \cdot k_2 \log(-z_{12}) + 2\alpha' k_2 \cdot k_3 \log(-z_{23})), \quad (5.2)$$

can be integrated by part to give

$$\begin{aligned} \delta A_4 &= - \int_0^1 dz_2 \partial_{z_2} \left(\frac{1}{z_{12}} \right) \exp(2\alpha' k_1 \cdot k_2 \log(-z_{12}) + 2\alpha' k_2 \cdot k_3 \log(-z_{23})) \\ &= \int_0^1 dz_2 \frac{1}{z_{12}} \partial_{z_2} \left(\exp(2\alpha' k_1 \cdot k_2 \log(-z_{12}) + 2\alpha' k_2 \cdot k_3 \log(-z_{23})) \right). \end{aligned} \quad (5.3)$$

By analytic continuation we can choose a kinematic region where the boundary terms vanish. Eq. 5.3 can be rewritten as

$$\delta A_4 = \int_0^1 dz_2 \frac{1}{z_{12}} \left(\frac{k_1 \cdot k_2}{z_{12}} + \frac{k_2 \cdot k_3}{z_{23}} \right) \left(\exp(2\alpha' k_1 \cdot k_2 \log(-z_{12}) + 2\alpha' k_2 \cdot k_3 \log(-z_{23})) \right), \quad (5.4)$$

where we recognize the 4-point scattering equation

$$S_2 = \frac{k_1 \cdot k_2}{z_{12}} + \frac{k_2 \cdot k_3}{z_{23}}. \quad (5.5)$$

Another ordering of the external legs will yield another scattering equation. The various ordered amplitudes are of course related by the action of the momentum kernel [12].

We see that in string theory we can trade the explicit $1/\alpha'$ -term by an integration over a term proportional to the scattering equation. In string theory this term of course gives a contribution.

The same phenomenon occurs for amplitudes with higher N . It gets increasingly tedious to carry out the sequence of partial integrations, but the origin of the mechanism seems to be closely related to a similar situation in string-based rules, proven in Appendix B of ref. [26] (see also ref. [18]). In this procedure, the last step is always a single integration by part on a variable that has been isolated, which, when the partial derivative hits the Koba-Nielsen factor, brings down a scattering equation in the integrand, just as in this four-point example.

Let us summarize the main point: We use integration by parts to rewrite the full string theory integrand. After having done these partial integrations, the new integrand now has the property that it

- (i) obviously reproduces the string theory amplitude to all orders in α' ,
- (ii) reproduces the field theory answer when $\alpha' = 0$ at the solution the scattering equations with the additional measure factor (2.6) .

This works because we have rewritten the string integrand as the Pfaffian plus terms proportional to the scattering equations. It illustrates again why the model corresponding to (ii) cannot be a string theory. Although we calculate the Pfaffian according to standard conformal field theory rules, the integrations by part of the $1/\alpha'$ -term is only a valid operation in the string theory integrand. When we subsequently in (ii) impose the $SL(2,C)$ invariant measure of scattering equations we can no longer interpret this as the same correlation function in two dimensions. However, it is *almost* a string theory, as we have seen: it yields the string theory amplitude in both limits $\alpha' \rightarrow 0$ and $\alpha' \rightarrow \infty$.

6. Amplitudes with Fermions and Mixed Amplitudes

6.1 The Four Fermion Amplitude

In this section we will show the generality of the delta function measure (2.6) by calculating a few tree-level amplitudes directly from string theory integrands. As a first example, we check how fermion amplitudes can come out from our prescription. In the case of the fermion four-point amplitude one has [23, 27]

$$\mathcal{A}_4 = \frac{g^2}{2} \alpha' \int dz z^{-2\alpha' t - 1} (1 - z)^{-2\alpha' s - 1} [(1 - z)(\gamma^\mu)_{\alpha_1 \alpha_2} (\gamma_\mu)_{\alpha_3 \alpha_4} - z(\gamma^\mu)_{\alpha_1 \alpha_4} (\gamma_\mu)_{\alpha_2 \alpha_3}] , \quad (6.1)$$

which is to be sandwiched between the external spinors $\bar{v}_{\alpha_1} u_{\alpha_2} \bar{v}_{\alpha_3} u_{\alpha_4}$. As is well known, this string theory integral can be done in terms of two beta functions. In the field theory limit $\alpha' \rightarrow 0$ it of course yields the correct answer corresponding to the two channels s and t .

But this integral also defines the correct field theory limit if we instead integrate over the δ -function measure given by the scattering equations as provided by the additional measure factor (2.6),

$$\mathcal{A}_4 = \frac{g^2}{2} \int dz \delta(S_2) z^{-2\alpha' t - 2} (1 - z)^{-2\alpha' s - 2} \times \\ ((1 - z)(\gamma^\mu)_{\alpha_1 \alpha_2} (\gamma_\mu)_{\alpha_3 \alpha_4} - z(\gamma^\mu)_{\alpha_1 \alpha_4} (\gamma_\mu)_{\alpha_2 \alpha_3}) , \quad (6.2)$$

where S_2 is the scattering equation in k_2 . Explicitly, we get in the limit $\alpha' \rightarrow 0$,

$$\mathcal{A}_4 = \frac{g^2}{2} \left[\frac{1}{s} (\gamma^\mu)_{\alpha_1 \alpha_2} (\gamma_\mu)_{\alpha_3 \alpha_4} - \frac{1}{t} (\gamma^\mu)_{\alpha_1 \alpha_4} (\gamma_\mu)_{\alpha_2 \alpha_3} \right] , \quad (6.3)$$

which is the correct field theory answer.

6.2 The Two Fermion Two Gluon Amplitude

As another example of how this procedure works one can similarly work out the expression for the two fermion two gluon amplitude. For the corresponding string theory integrand see, *e.g.*, refs. [23, 27]. This amplitude has also been considered in the ambitwistor framework of ref. [9], but here we simply get it for free from ordinary string theory.

We have explicitly verified in this case that the δ -function measure (2.6) yields exactly the tree-level amplitude in the limit $\alpha' \rightarrow 0$. In this case it follows in essentially one line, as there are no cancellations between tachyonic terms in the amplitudes. It indeed seems that we can directly take superstring integrands for amplitudes including fermions and integrate over a measure that localizes exactly on the scattering equations.

6.3 The Five Point Mixed Scalar-Gluon Amplitude

To give further credence to the procedure, let us finally consider a five-point case involving mixed external states of four scalars and a gluon. Because of the combination of scalars and a gluon, the string theory integrand of this amplitude contains two tachyonic terms cancelling each other in the integral, and we again first make this cancellation manifest by means of a single partial integration. We borrow the expression for the string theory integrand of the amplitude from ref. [28] (the explicit prefactor K_a in front of the integral can be found in that paper, but we do not need it for the arguments here),

$$\mathcal{A}_5(\phi_1, \phi_2, \phi_3, \phi_4, g_5) = K_a \int \left(\prod_{k=1}^5 dz_k \right) \left(\prod_{i < j} |z_{ij}|^{\alpha' s_{ij}} \right) \left(\frac{1}{z_{35}} \left(\frac{(\zeta_5 \cdot k_4)}{z_{45}} \frac{\alpha' s_{12} z_{34}}{z_{24} z_{13} z_{14} z_{23}} \right) \right. \\ \left. + \frac{(\zeta_5 \cdot k_1)}{z_{15} z_{24}} \left(\frac{(1 - \alpha' s_{24})}{z_{24} z_{13}} + \frac{\alpha' s_{24}}{z_{14} z_{23}} \right) \frac{(\zeta_5 \cdot k_2)}{z_{14} z_{25}} \left(\frac{(1 - \alpha' s_{14})}{z_{14} z_{23}} + \frac{\alpha' s_{14}}{z_{13} z_{24}} \right) \right) , \quad (6.4)$$

where $s_{ij} = k_i \cdot k_j$, ζ_i and k_i are the polarizations and momenta. Using the integration-by-parts relation in z_4 for the terms with ζ_5 dotted with k_1 and k_2 we can rewrite these explicit $1/\alpha'$ terms exactly as in the pure gluon case. This replaces that term by the scattering equation in leg 4, *e.g.* $(1 - \alpha' s_{14}) \rightarrow (\text{IBP}(S)_4 z_{14} - \alpha' s_{14})$. Using the prescription (2.6), we get

$$\begin{aligned} \mathcal{A}_5(\phi_1, \phi_2, \phi_3, \phi_4, g_5) = K_a \int \left(\prod_{k=1}^5 dz_k \right) & \left(\prod_{i < j} |z_{ij}|^{\alpha' s_{ij}} \right) \delta(S_4) \delta(S_5) \frac{1}{z_{23} z_{34} z_{45}} \\ & \left(\frac{1}{z_{35}} \left(\frac{(\zeta_5 \cdot k_4)}{z_{45}} \frac{\alpha' s_{12} z_{34}}{z_{24} z_{13} z_{14} z_{23}} \right) + \frac{(\zeta_5 \cdot k_1)}{z_{15} z_{24}} \left(\frac{\alpha' z_{24} S_4 - \alpha' s_{24}}{z_{24} z_{13}} + \frac{\alpha' s_{24}}{z_{14} z_{23}} \right) \right. \\ & \left. + \frac{(\zeta_5 \cdot k_2)}{z_{14} z_{25}} \left(\frac{\alpha' z_{14} S_4 - \alpha' s_{14}}{z_{14} z_{23}} + \frac{\alpha' s_{14}}{z_{13} z_{24}} \right) \right), \end{aligned} \quad (6.5)$$

where the δ -function measure now has been adapted to the situation where legs (1, 2, 3) are fixed as $(-\infty, 0, 1)$ following the convention used in [28]. We see that the δ -function effectively removes the $1/\alpha'$ term after having cancelled the tachyon pole explicitly by use of the partial integration that introduces the scattering equation in leg 4, $S_4 = 0$. After a little additional algebra we arrive in the limit of $(\alpha' \rightarrow 0)$ at

$$\mathcal{A}_5(\phi_1, \phi_2, \phi_3, \phi_4, g_5) = K_{ft} \left((\zeta_5 \cdot k_1) \left(\frac{1}{s_{23}} + \frac{s_{34}}{s_{23} s_{15}} \right) + (\zeta_5 \cdot k_2) \left(\frac{1}{s_{23}} \right) + (\zeta_5 \cdot k_4) \left(\frac{s_{12}}{s_{23} s_{45}} \right) \right), \quad (6.6)$$

which is the correct result. Here K_{ft} denotes the prefactor of the amplitude in the limit $(\alpha' \rightarrow 0)$. There thus seem to be no additional problems associated with mixed amplitudes. We therefore expect that any generic amplitude involving gluons, scalars and fermions in any combination can be computed in the same manner, imposing the same δ -function measure after having manifestly cancelled all tachyon poles (if present) through integrations by parts.

7. Conclusion

We have shown how to identify a unifying framework for string theory and scattering equations. It naturally leads to a new kind of dual model, string theory localized on the surface of the solutions to the scattering equations.

We believe this connection to the formalism of string theory is not fortuitous. To illustrate this, we have shown how to derive a new set amplitudes, those with external fermions, on the basis of merging string theory with the scattering equations. Numerous other examples can be derived similarly: mixed amplitudes gluons + fermions, scalars + fermions, scalars + fermions, and so on. We have provided some examples, and argued that the general prescription is to rewrite the string integrand so that it is manifestly free of tachyonic terms. This happens through partial derivatives which yield terms that vanish on the support of the scattering

equations. What is new here is that all needed integrands can be lifted directly from existing string theory computations, requiring only, generically, some partial integrations to manifestly cancel potential tachyon poles. No further calculations are necessary.

The modifications of the amplitudes at finite α' may have no significance, but we have kept them in because potentially they may find an interpretation. In string theory, this could perhaps be through a rigid minimal area constraint. In field theory, it looks like an unusual regularization factor that deforms the amplitude.

An obvious question is what happens at the genuine quantum level, *i.e.* at loop order. Tree level amplitudes correspond to vertex operators on the sphere. Using again string theory as the guide, one would be led to consider the corresponding scattering equations associated with the N external momenta but integrated over correlation functions on higher genus surfaces. Integrations will remain even after imposing the scattering equations. It would be interesting to see if they reproduce the result of field theory loop computations in the $\alpha' \rightarrow 0$ limit.

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A. Appendix: Further details on integration by parts.

In Section 5 we did not want to clutter the text with more explicit details of higher-point issues with respect to the needed integration by parts. In this Appendix we provide a few details of what happens at five points.

To illustrate in this slightly more complicated case how to do the integration by parts, we will consider the term originating from the $\partial X(z_i)\partial X(z_j)$ contractions in the picture changing formalism. As in the four-point case we choose to remove lines 1 and 2 in the matrix of the Pfaffian.

Explicitly, we have the following two types of terms the are of type $\frac{1}{\alpha'}$

$$\begin{aligned} \sim \dots & - \frac{\epsilon_1 \cdot k_3 \epsilon_2 \cdot \epsilon_3 \epsilon_4 \cdot \epsilon_5}{\alpha' z_{12} z_{13} z_{23} z_{45}^2} + \frac{\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot k_3 \epsilon_4 \cdot \epsilon_5}{\alpha' z_{12} z_{13} z_{23} z_{45}^2} - \frac{\epsilon_1 \cdot k_4 \epsilon_2 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_5}{\alpha' z_{12} z_{14} z_{24} z_{35}^2} \\ & + \frac{\epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot k_4 \epsilon_3 \cdot \epsilon_5}{\alpha' z_{12} z_{14} z_{24} z_{35}^2} - \frac{\epsilon_1 \cdot k_5 \epsilon_2 \cdot \epsilon_5 \epsilon_3 \cdot \epsilon_4}{\alpha' z_{12} z_{15} z_{25} z_{34}^2} + \frac{\epsilon_1 \cdot \epsilon_5 \epsilon_2 \cdot k_5 \epsilon_3 \cdot \epsilon_4}{\alpha' z_{12} z_{15} z_{25} z_{34}^2}, \end{aligned} \quad (\text{A.1})$$

and

$$\begin{aligned}
& \sim \dots - \frac{\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 \left(\frac{\epsilon_5 \cdot k_1}{z_{15}} + \frac{\epsilon_5 \cdot k_2}{z_{25}} + \frac{\epsilon_5 \cdot k_3}{z_{35}} + \frac{\epsilon_5 \cdot k_4}{z_{45}} \right)}{\alpha' z_{12}^2 z_{34}^2} \\
& - \frac{\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_5 \left(\frac{\epsilon_4 \cdot k_1}{z_{14}} + \frac{\epsilon_4 \cdot k_2}{z_{24}} + \frac{\epsilon_4 \cdot k_3}{z_{34}} - \frac{\epsilon_4 \cdot k_5}{z_{45}} \right)}{\alpha' z_{12}^2 z_{35}^2} + \frac{\epsilon_1 \cdot \epsilon_2 \epsilon_4 \cdot \epsilon_5 \left(-\frac{\epsilon_3 \cdot k_1}{z_{13}} - \frac{\epsilon_3 \cdot k_2}{z_{23}} + \frac{\epsilon_3 \cdot k_4}{z_{34}} + \frac{\epsilon_3 \cdot k_5}{z_{35}} \right)}{\alpha z_{12}^2 z_{45}^2}.
\end{aligned} \tag{A.2}$$

We will now show that in all cases we can find integration by part relations that are equivalent to inserting the scattering equations.

- In the first equation (A.1) we will in terms 1 - 2 use the relation involving z_4 , while for the terms 3 - 6 we will instead use the integration by part relation in z_3 .
- In the second equation (A.2) for the first term we will use the relation in the variable z_3 , except for the next-to-last term where we will use the one for z_4 . For the second and third terms here we will use those of in z_1 , except for the first terms where we use those of in z_3 and z_4 .

By this prescription we have absorbed all the $\frac{1}{\alpha'}$ terms of the five-point amplitude. Again we observe that at the solution to the scattering equations the reduced Pfaffian will be unchanged, since all we have done is to turn them into terms proportional to the scattering equations.

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